

# Ferromagnetic properties of quark matter - an origin of magnetic field in compact stars -

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## 1 Introduction

The phase diagram of QCD in the density ( $\rho_B$ ) - temperature ( $T$ ) plane has been explored by many authors; QGP in high- $T$  region or color superconductivity in high- $\rho_B$  region is a typical phase in that plane. Here we are interested in low temperature and moderate density region relevant to compact stars, where magnetic order is expected.

Origin of the magnetic field in compact stars is one of the long-standing problems since the first discovery of pulsars in early seventies. Recent discovery of magnetars with huge magnetic field of  $O(10^{14-15}\text{G})$  has revived this problem. Since nuclear matter is developed inside compact stars, we are tempted to consider spontaneous spin polarization of nucleons as a microscopic origin of the magnetic field. Realistic calculations have been performed for polarized nuclear matter, but they lead us to the negative results [1]. In ref.[2] possibility of spin polarization of quark matter has been suggested by a simple consideration in analogy with itinerant electrons, where the Fock exchange interaction is responsible to ferromagnetism [3]. A weakly first-order phase transition has been demonstrated around nuclear density. Using this result we can roughly estimate the magnitude of the magnetic field at the surface to be  $O(10^{15-17}\text{G})$ , which may explain the magnetic field of magnetars. The coexistence of ferromagnetism with color superconductivity has been also discussed in ref. [4].

Here we apply the Landau Fermi-liquid theory (FLT) to elucidate the critical behavior of the magnetic phase transition at finite density and temperature [5]. We evaluate the magnetic susceptibility of quark matter. The divergence and sign change of the magnetic susceptibility is a signal of the magnetic instability to the ferromagnetic phase, since its inverse measures the curvature of the free energy at the origin with respect to the magnetization. Thus quarks near the Fermi surface are responsible to the magnetic transition and the spin dependent quark-quark interaction and the density of states near the Fermi surface are the key ingredients within FLT.

Theoretically we find a non-Fermi-liquid behavior of the magnetic susceptibility. It is well known that there appears a non-Fermi-liquid behavior in the expression of the specific heat in QCD as well as QED, which is caused by the transverse gauge field because it is not statically screened [6].

## 2 Relativistic Fermi liquid theory

Within the Landau Fermi-liquid theory (FLT) we assume a one-to-one correspondence between the states of the free Fermi gas and those of the interacting system [5]. Quarks are treated as quasi-particles carrying the same quantum numbers of the free quarks, and the quasi-particle distribution function is simply given by the Fermi-Dirac one,

$$n(\mathbf{k}, \zeta) = [1 + \exp(\beta(\epsilon_{\mathbf{k}, \zeta} - \mu))]^{-1} \quad (1)$$

with the quasi-particle energy  $\epsilon_{\mathbf{k}, \zeta}$  specified by the momentum  $\mathbf{k}$  and a spin quantum number  $\zeta = \pm 1$ .

### 2.1 Screening effect on the quasi-particle interaction

In the following we consider the color-symmetric interaction among quasi-particles that can be written as the sum of two parts, the spin independent ( $f_{\mathbf{k}, \mathbf{q}}^s$ ) and dependent ( $f_{\mathbf{k}, \mathbf{q}}^a$ ) terms;

$$f_{\mathbf{k}\zeta, \mathbf{q}\zeta'} = f_{\mathbf{k}, \mathbf{q}}^s + \zeta\zeta' f_{\mathbf{k}, \mathbf{q}}^a. \quad (2)$$

Since quark matter is color singlet as a whole, the Fock exchange interaction gives a leading contribution. We, hereafter, consider the one-gluon-exchange interaction (OGE). For a pair with color index  $(a, b)$ , the Fock exchange interaction gives a factor  $(\lambda_\alpha)_{ab}(\lambda_\alpha)_{ba} = 1/2 - 1/(2N_c)\delta_{ab}$ , which is always positive for any pair. Hence the situation is very similar to electron gas in QED. Since we are interested in the electromagnetic properties of quark matter, only the color symmetric interaction is relevant, which is written as

$$f_{\mathbf{k}\zeta, \mathbf{q}\zeta'} = \frac{1}{N_c^2} \sum_{a, b} f_{\mathbf{k}\zeta a, \mathbf{q}\zeta' b} = \frac{m}{E_k} \frac{m}{E_q} M_{\mathbf{k}\zeta, \mathbf{q}\zeta'}, \quad (3)$$

with the invariant matrix element,

$$M_{\mathbf{k}\zeta, \mathbf{q}\zeta'} = -g^2 \frac{1}{N_c^2} \text{tr} (\lambda_\alpha/2 \lambda_\alpha/2) M^{\mu\nu}(k, \zeta; q, \zeta') D_{\mu\nu}(k - q), \quad (4)$$

where  $M^{\mu\nu}(k, \zeta; q, \zeta') = \text{tr} [\gamma^\mu \rho(k, \zeta) \gamma^\nu \rho(q, \zeta')]$ .

Since the OGE interaction is a long-range force and we consider the small energy-momentum transfer between quasi-particles, we must treat the gluon propagator by

taking into account HDL resummation. Thus we take into account the screening effect,

$$D_{\mu\nu}(k-q) = P_{\mu\nu}^t D_t(p) + P_{\mu\nu}^l D_l(p) - \xi \frac{p_\mu p_\nu}{p^4} \quad (5)$$

with  $p = k - q$ , where  $D_{t(l)}(p) = (p^2 - \Pi_{t(l)})^{-1}$ , and the last term represents the gauge dependence with a parameter  $\xi$ .  $P_{\mu\nu}^{t(l)}$  is the projection operator onto the transverse (longitudinal) mode,

$$\begin{aligned} P_{\mu\nu}^t &= (1 - g_{\mu 0})(1 - g_{\nu 0}) \left( -g_{\mu\nu} - \frac{p_\mu p_\nu}{|\mathbf{p}|^2} \right) \\ P_{\mu\nu}^l &= -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} - P_{\mu\nu}^t. \end{aligned} \quad (6)$$

The self-energies for the transverse and longitudinal gluons are given as

$$\begin{aligned} \Pi_l(p_0, \mathbf{p}) &= \sum_{f=u,d,s} \left( m_{D,f}^2 + i \frac{\pi m_{D,f}^2}{2u_{F,f}} \frac{p_0}{|\mathbf{p}|} \right) \\ \Pi_t(p_0, \mathbf{p}) &= -i \sum_{f=u,d,s} \frac{\pi u_{F,f} m_{D,f}^2}{4} \frac{p_0}{|\mathbf{p}|}, \end{aligned} \quad (7)$$

in the limit  $p_0/|\mathbf{p}| \rightarrow 0$ , with  $u_{F,f} \equiv k_{F,f}/E_{F,f}$  and the Debye mass for each flavor,  $m_{D,f}^2 \equiv g^2 \mu_f k_{F,f}/2\pi^2$ . Thus the longitudinal gluons are statically screened to have the Debye mass, while the transverse gluons are dynamically screened by the Landau damping, in the limit  $p_0/|\mathbf{p}| \rightarrow 0$ . Accordingly, the screening effect for the transverse gluons is ineffective at  $T = 0$ , where soft gluons ( $p_0/|\mathbf{p}| \rightarrow 0$ ) contribute. At finite temperature, gluons with  $p_0 \sim O(T)$  can contribute due to the diffuseness of the Fermi surface and the transverse gluons are effectively screened.

## 2.2 Magnetic susceptibility

We consider the linear response of the normal(unpolarized) quark matter by applying a small magnetic field  $\mathbf{B}$ . Using the Gordon identity, the coupling term with the uniform magnetic field ( $\mathbf{A} = \mathbf{B} \times \mathbf{r}/2$ ) can be written as

$$\begin{aligned} \int d^4x \mathcal{L}_{\text{int}} &= e_q \int d^4x \bar{q} \boldsymbol{\gamma} \cdot \mathbf{A} q \\ &= \mu_q \int d^4x \bar{q} [\mathbf{L} + \boldsymbol{\Sigma}] \cdot \mathbf{B} q, \end{aligned} \quad (8)$$

with  $\mu_q$  being the Dirac magneton. We discard the contribution of the orbital angular momentum  $\langle \mathbf{L} \rangle$  by assuming the uniform distribution of quarks. Thus the magnetization  $\langle M \rangle_f$  for each flavor can be written as

$$\langle M \rangle_f = V^{-1} \left\langle \int d^3x \bar{q}_f \Sigma_z q_f \right\rangle, \quad (9)$$

where we take  $\mathbf{B}/\hat{\mathbf{z}}$ . Accordingly the magnetic susceptibility is defined as

$$\chi_M = \sum_{f=u,d,s} \chi_M^f = \sum_{f=u,d,s} \left. \frac{\partial \langle M \rangle_f}{\partial B} \right|_{B=0}. \quad (10)$$

Hereafter, we shall concentrate on one flavor and omit the flavor indices because the magnetic susceptibility is given by the sum of the contribution from each flavors. The magnetic susceptibility is proportional to the number difference between different spin states ( $\zeta = \pm 1$ ); it is explicitly caused by the applying magnetic field, and implicitly caused through the spin-dependent interaction,

$$\delta n_{\mathbf{k}\zeta=+1} - \delta n_{\mathbf{k}\zeta=-1} = \frac{\partial n_{\mathbf{k}}}{\partial \epsilon_{\mathbf{k}}} [-g_D \mu_q B + \delta \epsilon_{\mathbf{k}\zeta=+1} - \delta \epsilon_{\mathbf{k}\zeta=-1}] \quad (11)$$

with the gyromagnetic ratio  $g_D \sim 2$ , where

$$\delta \epsilon_{\mathbf{k}\zeta} = N_c \sum_{\zeta'=\pm 1} \int \frac{d^3 q}{(2\pi)^3} f_{\mathbf{k}\zeta;\mathbf{q}\zeta'} \delta n_{\mathbf{q}\zeta'}. \quad (12)$$

Magnetic susceptibility is then written in terms of the quasi-particle interaction,

$$\chi_M = \left( \frac{\bar{g}_D \mu_q}{2} \right)^2 \frac{N(T)}{1 + N(T) \bar{f}^a} \quad (13)$$

where  $\bar{g}_D$  is an angle average of  $g_D$ , and  $\bar{f}^a$  is the Landau-Migdal parameter averaged over the Fermi surface [11, 12].

### 3 Magnetic properties at $T = 0$

$N(T)$  is the effective density of states at the Fermi surface, and is simply written as

$$N^{-1}(0) = \frac{\pi^2}{N_c k_F^2} v_F \quad (14)$$

in the limit of zero temperature. Eq. (14) defines the Fermi velocity, which is given by using the Lorentz transformation [5],

$$v_F \equiv \left. \frac{\partial n_{\mathbf{k}}}{\partial \epsilon_{\mathbf{k}}} \right|_{|\mathbf{k}|=k_F} = \frac{k_F}{\mu} - \frac{N_c k_F^2}{3\pi^2} f_1^s, \quad (15)$$

where  $f_1^s$  is a spin-averaged Landau-Migdal parameter.

Finally the magnetic susceptibility at zero temperature can be written in terms of the Landau-Migdal parameters,

$$\chi_M = \chi_{\text{Pauli}} \left[ 1 + \frac{N_c k_F \mu}{\pi^2} \left( -\frac{1}{3} f_1^s + \bar{f}^a \right) \right]^{-1}, \quad (16)$$

where  $\chi_{\text{Pauli}}$  is the usual one for the Pauli paramagnetism,  $\chi_{\text{Pauli}} = \bar{g}_D^2 \mu_q^2 N_c k_F \mu / 4\pi^2$ .

The quasiparticle interaction on the Fermi surface can be written as

$$f_{\mathbf{k}\zeta, \mathbf{q}\zeta'}|_{|\mathbf{k}|=|\mathbf{q}|=k_F} = -C_g \frac{m^2}{E_F^2} \left[ -M^{00} D_L(\mathbf{k} - \mathbf{q}) + M^{ii} D_T(\mathbf{k} - \mathbf{q}) \right], \quad (17)$$

with the effective coupling strength,  $C_g = \frac{N_c^2 - 1}{2N_c^2} g^2$ .

We can see that the both Landau parameters  $f_1^s, \bar{f}^a$  include the infrared singularities due to the absence of the static screening for the transverse gluons;  $D_T(\mathbf{k} - \mathbf{q}) \sim -1/(\mathbf{k} - \mathbf{q})^2 = -1/2k_F^2(1 - \cos\theta_{\mathbf{k}\mathbf{q}})$  in this case, so that the logarithmic divergences appear in the Landau parameters through the integral over the relative angle,  $\int d\Omega_{\mathbf{k}\mathbf{q}} 1/(1 - \cos\theta_{\mathbf{k}\mathbf{q}})$ .

Finally magnetic susceptibility is given as a sum of the contributions of the bare interaction and the static screening effect. We can see that the logarithmic divergences exactly cancel each other to give a finite result for susceptibility [8, 9].

$$(\chi_M/\chi_{\text{Pauli}})^{-1} = 1 - \frac{C_g N_c \mu}{12\pi^2 E_F^2 k_F} \left[ m(2E_F + m) - \frac{1}{2}(E_F^2 + 4E_F m - 2m^2) \kappa \ln \frac{2}{\kappa} \right], \quad (18)$$

with  $\kappa = m_D^2/2k_F^2$ . Obviously this expression is reduced to the simple OGE case without screening in the limit  $\kappa \rightarrow 0$ ; one can see that the interaction among massless quarks gives a null contribution for the magnetic transition. The effect of the static screening for the longitudinal gluons gives the contribution of  $g^4 \ln(1/g^2)$ . In the nonrelativistic limit, it recovers the corresponding term in the RPA calculation of electron gas [3, 8, 9].

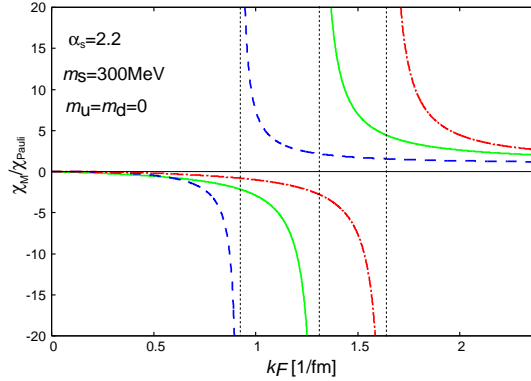


Figure 1: Magnetic susceptibility at  $T = 0$ . The solid curve shows the result using simple OGE, while the dashed and dash-dotted ones show the screening effects with  $N_f = 1$  (only s quarks) and with  $N_f = 3$  (u, d, and s quarks) respectively.

In Fig. 1, we plot the magnetic susceptibility at  $T = 0$  [8, 9]. We take the QCD coupling constant as  $\alpha_s \equiv g^2/4\pi = 2.2$  and the strange quark mass  $m_s = 300\text{MeV}$  inferred from the MIT bag model. We consider here the MIT bag model as an effective model succeeded in reproducing the low-lying hadron spectra. The coupling constant looks rather large, but this value is required for the color magnetic interaction to explain the mass splitting of hadrons with different spins; e.g. for nucleon and  $\Delta$  isobar. We think this feature is relevant in our study, because the coupling constant is closely related to the strength of the spin-spin interaction between quarks in this model. Moreover, the quark density in the MIT bag model is moderate,  $0.25\text{fm}^{-3}$ , which is the similar one we are interested in. Note that the perturbation method should be still meaningful even for this rather large coupling, since the renormalization-group analysis has shown that the relevant expansion parameter is not the gauge coupling constant  $g^2$  but the product of  $g^2$  with the Fermi velocity  $v_F$ , which *always* goes to zero as one approaches to the Fermi surface [7].

One can see that the magnetic susceptibility for the simple OGE without screening diverges around  $k_F = 1.3\text{ fm}^{-1}$ . This is consistent with the previous result for the energy calculation. [8, 9]. One may expect that the screening effect weakens the Fock exchange interaction so that the critical density get lower once we take into account the screening effect. However, this is not necessarily the case in QCD. The screening effect behaves in different ways depending on the number of flavors. Compare the results for the  $N_f = 3$  with the one for  $N_f = 1$ . In the case of  $N_f = 1, \kappa \leq 2$  the screening effect works against the magnetic phase transition as in QED. However, for  $N_f = 3, \kappa > 2$  so that the critical density is increased. Consequently the screening effect does not necessarily work against the magnetic instability, which is a different aspect from electron gas [9].

## 4 Finite temperature effects and Non-Fermi-liquid behavior

At finite temperature, the magnetic susceptibility is given by

$$\chi_M = \left( \frac{\bar{g}_D \mu_q}{2} \right)^2 \left[ N^{-1}(T) + \bar{f}_l^a + \bar{f}_t^a \right]^{-1} \quad (19)$$

where  $\bar{f}_l^a$  and  $\bar{f}_t^a$  denote the longitudinal and transverse parts of  $\bar{f}^a$  respectively [11, 12]. First, we evaluate the effective density of states on the Fermi surface defined by

$$N(T) = \frac{N_c}{\pi^2} \int_{\epsilon_0}^{\infty} d\omega \frac{dk}{d\omega} k^2 \frac{\beta e^{\beta(\omega-\mu)}}{(e^{\beta(\omega-\mu)} + 1)^2}, \quad (20)$$

with  $\epsilon_0 \equiv \epsilon_{|\mathbf{k}|=0}$ . The quasi-particle energy  $\omega$  should be given as a solution of the equation,

$$\omega = E_{k(\omega)} + \text{Re}\Sigma_+(\omega, k(\omega)), \quad (21)$$

where we discard the imaginary part within the quasi-particle approximation.

The one-loop self-energy is almost independent of the momentum, and can be written as [10]

$$\text{Re}\Sigma_+(\omega, k) \sim \text{Re}\Sigma_+(\mu, k_F) - \frac{C_f g^2 u_F}{12\pi^2} (\omega - \mu) \ln \frac{\Lambda}{|\omega - \mu|} + \Delta^{\text{reg}}(\omega - \mu) \quad (22)$$

around  $\omega \sim \mu$  with  $C_f = (N_c^2 - 1)/(2N_c)$  and  $u_F = k_F/E_{k_F}$ .  $\Lambda$  is a cut-off factor and should be an order of the Debye mass,  $\Lambda \sim O(m_D)$ . Note that the anomalous term in Eq. (22) appears from the dynamic screening of the transverse gluons, and the contribution by the longitudinal gluons is summarized in the regular function  $\Delta^{\text{reg}}(\omega - \mu)$  of  $O(g^2)$ . Within the approximation given by Eqs. (22) and (??), the self-energy is independent of spatial momentum  $k$  and thus we omit the argument  $k$  hereafter. The renormalization factor  $z_+(k)$  is then given by the equation,  $z_+(k) = (1 - \partial \text{Re}\Sigma_+(\omega)/\partial \omega|_{\omega=\epsilon_{\mathbf{k}}})^{-1}$ , and we have

$$z_+(k)^{-1} \sim -\frac{C_f g^2 u_F}{12\pi^2} \ln |\epsilon_{\mathbf{k}} - \mu|. \quad (23)$$

It exhibits a logarithmic divergence as  $\epsilon_{\mathbf{k}} \rightarrow \mu$ , which causes non-Fermi liquid behavior [7].

Eventually,  $N(T)$  is written as,

$$N(T) \simeq \frac{N_c}{\pi^2} \int_{\epsilon_0}^{\infty} d\omega \left( 1 - \frac{\partial \text{Re}\Sigma_+(\omega)}{\partial \omega} \right) k(\omega) E_{k(\omega)} \frac{\beta e^{\beta(\omega-\mu)}}{(e^{\beta(\omega-\mu)} + 1)^2}. \quad (24)$$

We can separate the contribution by the longitudinal gluons  $N_l(T)$  from  $N(T)$ . Since the longitudinal gluon exchange is short-ranged by the Debye screening mass, it becomes almost temperature independent,

$$N_l(T) = \frac{N_c k_F E_F}{3\pi^2} f_{l;1}^s, \quad (25)$$

with the Landau-Migdal parameter  $f_{l;1}^s$ ,

$$f_{l;1}^s = -\frac{3N_c^{-1} C_f g^2}{8E_F^2 k_F^2} [\kappa k_F^2 + 2E_F^2] [(1 + \kappa)I_0(\kappa) - 1], \quad (26)$$

where  $\kappa = \sum_f m_{D,f}^2/2k_F^2$  and

$$I_0(\kappa) = \frac{1}{2} \int_{-1}^1 \frac{du}{1 - u + \kappa} \simeq \frac{1}{2} \ln \left( \frac{2}{\kappa} \right) \simeq \ln(g^{-2}). \quad (27)$$

To evaluate the transverse contribution,  $N_t(T) = N(T) - N_l(T)$ , we only use the transverse part in Eq. (22): substituting Eq. (22) into Eq. (24), we obtain the leading order contribution<sup>1</sup>,

$$N_t(T) = \frac{N_c k_s \mu}{\pi^2} \left[ 1 + \frac{\pi^2 (2k_F^2 - m^2)}{6 k_F^4} T^2 + \frac{C_f g^2 u_F (2k_F^2 - m^2)}{24 k_F^4} T^2 \ln \left( \frac{\Lambda}{T} \right) + \frac{C_f g^2 u_F}{12\pi^2} \ln \left( \frac{\Lambda}{T} \right) \right] + O(g^2 T^2), \quad (28)$$

after some manipulation.  $N_t(T)$  has a term proportional to  $\ln T$ , which gives a singularity at  $T = 0$ . This singularity corresponds to the logarithmic divergence of the Landau-Migdal parameter  $f_1^s$  at  $T = 0$ . The chemical potential  $\mu$  in Eq. (28) implicitly includes the temperature dependence. To extract the proper temperature dependence in  $\chi_M$  we must carefully take into account the temperature dependence of  $\mu$ . Using the thermodynamic relation  $\mu = -(\partial F / \partial n)|_T$  with the free energy  $F = E - Ts$ , we have [12]

$$\mu(T) = \mu_0 - \frac{\pi^2}{6} \frac{2k_F^2 + m^2}{k_F^2 E_F} T^2 \left( 1 + \frac{C_f g^2 u_F}{12\pi^2} \ln \left( \frac{\Lambda}{T} \right) \right) + O(g^2 T^2). \quad (29)$$

We can see that  $\mu$  includes  $T^2 \ln T$  term due to the dynamic screening effect for the transverse gluons, besides the usual  $T^2$  term.

As for the spin-dependent Landau-Migdal parameter, the leading-order contribution at finite temperature comes from the transverse component  $\bar{f}_t^a$ ; it has a logarithmic singularity at  $T = 0$  due to the dynamic screening effect. In this section, we shall see that the logarithmic divergences of  $N^{-1}(T)$  and  $\bar{f}_t^a$  at  $T = 0$  cancel out each other to give a finite contribution to the magnetic susceptibility.  $\bar{f}_t^a$  is given by

$$\bar{f}_t^a = -2N_c N^{-1}(T) \int \frac{d^3 k}{(2\pi)^3} \frac{\partial n(\epsilon_{\mathbf{k}})}{\partial \epsilon_{\mathbf{k}}} \bar{f}_{t;k,k_s}^a \quad (30)$$

with

$$\bar{f}_{t;k,k_s}^a = - \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \int \frac{d\Omega_{\mathbf{q}}}{4\pi} \frac{m^2}{E_s E_{\mathbf{k}}} C_f N_c^{-1} g^2 M^{iia} D_t(k - q) \Big|_{|\mathbf{q}|=k_s} \quad (31)$$

where  $M^{iia}$  is the spin-dependent component of  $M^{ii}$  in Eq.(4), and  $k_s = k_F + O(T^2)$  is defined by  $\epsilon_{k_s} = \mu$ .

The real part of the transverse propagator is

$$\text{Re} D_t(k - q) \Big|_{|\mathbf{q}|=k_s} = \frac{(k - q)^2}{\{(k - q)^2\}^2 + \left( \frac{1}{4} \sum_f \pi u_{F,f} m_{D,f}^2 \right)^2 \frac{(E_{\mathbf{k}} - E_s)^2}{(\mathbf{k} - \mathbf{q})^2}} \Big|_{|\mathbf{q}|=k_s} \quad (32)$$

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<sup>1</sup>We discard here the temperature independent term of  $O(g^2)$ , which cannot be given only by Eq. (22). However, we can recover it by taking the  $T \rightarrow 0$  limit later.



, while the imaginary part gives only a sub-leading contribution and can be discarded.

The integral over  $k$  in Eq. (30) can be performed as in Eq. (21). Finally we find a leading-order contribution at  $T \neq 0$ ,

$$\begin{aligned}\bar{f}_t^a &\sim N^{-1}(T) \frac{C_f g^2}{12\pi^2 k_s E_s} \left[ 1 + \frac{\pi^2 (2k_s^2 - m^2)}{6 k_s^4} T^2 \right] \ln T^{-1} + O(g^2 T^2) \\ &\sim \frac{C_f g^2}{12N_c E_s \mu} \ln T^{-1}.\end{aligned}\tag{33}$$

Compare Eq. (33) with Eq. (27). Since  $E_s = E_F + O(T^2)$  and  $k_s = k_F + O(T^2)$  as we shall see, the  $\ln T$  terms cancel each other in the magnetic susceptibility (19).

$$\begin{aligned}(\chi_M/\chi_{\text{Pauli}})^{-1} &= 1 - \frac{C_f g^2}{12\pi^2 E_F k_F} \left[ m(2E_F + m) - \frac{1}{2}(E_F^2 + 4E_F m - 2m^2)\kappa \ln \frac{2}{\kappa} \right] \\ &+ \frac{\pi^2}{6k_F^4} \left( 2E_F^2 - m^2 + \frac{m^4}{E_F^2} \right) T^2 + \frac{C_f g^2 u_F}{72} \frac{(2k_F^4 + k_F^2 m^2 + m^4)}{k_F^4 E_F^2} T^2 \ln \left( \frac{\Lambda}{T} \right) \\ &+ O(g^2 T^2).\end{aligned}\tag{34}$$

In Fig.2, we plot the magnetic susceptibility given by Eq. (34). At  $T=0$ , the magnetic susceptibility is positive at higher densities and the quark matter is in the paramagnetic phase there. At the critical density where the magnetic susceptibility diverges ( $k_F^c \sim 1.6\text{fm}^{-1}$ ), there occurs a magnetic phase transition from the paramagnetic phase to the ferromagnetic phase and the quark matter remains in the ferromagnetic phase below  $k_F^c$ .

At  $T=30$  MeV, there appear two critical densities at which the magnetic susceptibility diverges. We denote these densities  $k_{F1}^c$  and  $k_{F2}^c$  ( $k_{F1}^c < k_{F2}^c$ ). In this case,  $k_{F1}^c \simeq 0.4\text{fm}^{-1}$  and  $k_{F2}^c \simeq 1.5\text{fm}^{-1}$ . At densities below  $k_{F1}^c$  and above  $k_{F2}^c$ , the magnetic susceptibility is positive, which corresponds to the paramagnetic phase, on the other hand, at densities between two critical densities, it becomes negative corresponding to the ferromagnetic phase.

At  $T=50$  MeV, there are still two critical densities ( $k_{F1}^c \simeq 0.7\text{fm}^{-1}$  and  $k_{F2}^c \simeq 1.3\text{fm}^{-1}$ ), but the range between these two densities becomes narrower than at  $T=30$  MeV.

At  $T=60$  MeV, there is no longer divergence in the magnetic susceptibility and quark matter is in the paramagnetic phase at any density.

We show a magnetic phase diagram of QCD on the density-temperature plane in Fig.3. The four curves corresponds to the critical curves given by Eq.(34) under four different assumptions: below the curves the quark matter is in the ferromagnetic phase, while it is in the paramagnetic phase above the critical curves. The magnetic transition occurs on the critical curves.

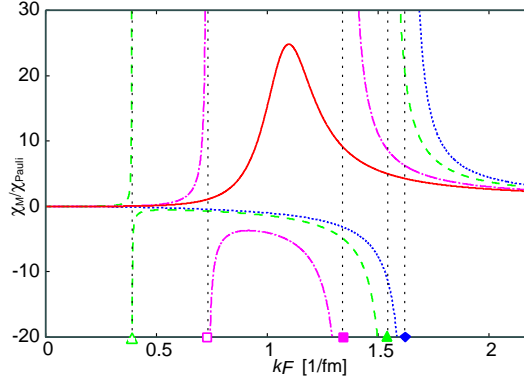


Figure 2: Magnetic susceptibility at finite temperature. The dotted, dashed, dash-dotted, and solid curves show the results at  $T=0, 30, 50$ , and  $60$  MeV respectively.

For the solid curve, we have used the full expression Eq.(34), on the other hand, for the dashed, dash-dotted, and dotted curves, we have ignored the dynamic screening(*i.e.* the  $T^2 \ln T$  term), static screening(*i.e.* the  $\kappa \ln \kappa$  term), and both of the two screenings in Eq. (34) respectively.

Compare the result with the full expression (34) with the one without the non-Fermi-liquid effect *i.e.*  $T^2 \ln T$  dependence. In the case without the  $T^2 \ln T$  term, the ferromagnetic phase can be sustained till over  $T = 60$  MeV, while it can be at most  $T = 60$  MeV including  $T^2 \ln T$  dependence. It turns out that the dynamic screening works against the magnetic instability and can reduce the ferromagnetic region in the phase diagram up to a point, but this effect is not so large.

The dash-dotted curve is the result without the static screening or  $\kappa \ln \kappa$  term in Eq.(34). The static screening effect works in favor of the magnetic instability to enlarge the ferromagnetic region. As discussed in [9], it depends on the number of flavors whether the static screening works for the ferromagnetism or not, which is peculiar to QCD.

The maximum Curie temperature  $T_c^{\max}$  is around 60 MeV, which is achieved at  $k_F \simeq 1.1 \text{ fm}^{-1}$ . Note that this is still low temperature, since  $T_c^{\max}/k_F \ll 1$ . Thus our low-temperature expansion is legitimate over all points on the critical curve. One of the interesting phenomenological implications may be related to thermal evolution of magnetars; during the supernova expansions temperature rises up to several tens MeV, which is so that ferromagnetic phase transition may occur in the initial cooling stage to produce huge magnetic field.

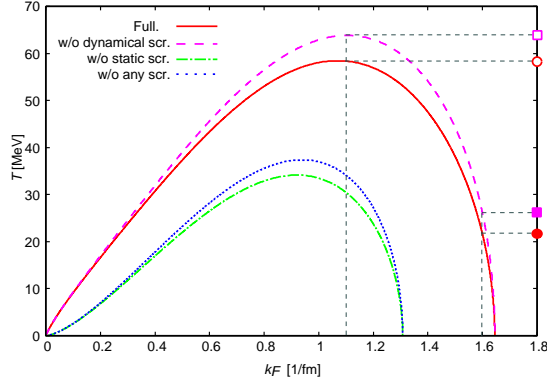


Figure 3: Magnetic phase diagram in the density-temperature plane. The solid, dashed, dash-dotted, dotted curves show the results for the full expression Eq. (34), the one without the  $T^2 \ln T$  term, without the  $\kappa \ln \kappa$  term, and without the  $T^2 \ln T$  and  $\kappa \ln \kappa$  terms in Eq. (34). The open (filled) circle indicates the Curie temperature at  $k_F = 1.1(1.6)$   $\text{fm}^{-1}$  while the squares show those when we disregard the  $T^2 \ln T$  dependence.

## 5 Outlook

We have discussed the critical behavior of the magnetic susceptibility in the density-temperature plane within the Fermi liquid theory. We have found a novel non-Fermi-liquid behavior and phase boundary by a perturbative calculations. Some non-perturbative effects such as instanton effects should be taken into account at moderate densities. This is important not theoretically but also phenomenologically; more realistic estimate of the critical density or the Curie temperature is needed when we face phenomena in compact stars.

There are various ideas such as amplification of the fossil field for the origin of the magnetic field in compact stars. So it should be very interesting if we can distinguish these ideas through observations. To this end we must consider not only magnetic evolution but also thermal evolution; if ferromagnetic state is realized, spin waves should be excited which affect the thermal evolution of compact stars [8].

This work was partially supported by the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan and the Grant-in-Aid for Scientific Research (C) (20540267).

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